RELATIVITY AND COSMOLOGY I

Solutions to Problem Set 10 Fall 2022

1. The Precession of Mercury

The solutions to this problem and the phrasing of the problem itself were based on the paper "Simple precession calculation for Mercury: A linearization approach" by Michael J.W. Hall, published in Am.J.Phys. 90 (2022) 11, 857, Am.J.Phys. 90 (2022) 857-860.

(a) In Newtonian gravity, the total energy is

$$\mathcal{E} = \frac{mv^2}{2} + V = \frac{m\dot{r}^2}{2} + \frac{mr^2\dot{\phi}^2}{2} - \frac{GMm}{r}\,.$$

Using the definition of angular momentum $\mathcal{L} = mr^2\dot{\phi}$, one can write

$$\mathcal{E} = \frac{m\dot{r}^2}{2} + \frac{\mathcal{L}^2}{2mr^2} - \frac{r_s m}{2r} \tag{1}$$

We can then obtain the required form:

$$\dot{r}^2 = \frac{2\mathcal{E}}{m} + \frac{r_s}{r} - \frac{\mathcal{L}^2}{m^2r^2} \tag{2}$$

Now we use

$$\frac{dr}{dt} = \frac{d\phi}{dt}\frac{dr}{d\phi} = \frac{\mathcal{L}}{mr^2}\frac{dr}{d\phi} = -\frac{\mathcal{L}}{m}\frac{d}{d\phi}\left(\frac{1}{r}\right) = -\frac{\mathcal{L}}{m}\frac{du}{d\phi}\,. \tag{3}$$

The equation becomes

$$\frac{\mathcal{L}^2}{m^2}\left(\frac{du}{d\phi}\right)^2 = \frac{2\mathcal{E}}{m} + r_s u - \frac{\mathcal{L}^2}{m^2}u^2\,. \tag{4}$$

Differentiating both sides, we get

$$\frac{d^2u}{d\phi^2} + u = \frac{m^2}{\mathcal{L}^2}\frac{r_s}{2}\,. \tag{5}$$

(b) The metric only depends on r and θ, therefore t and ϕ are cyclic coordinates, with

$$E = g_{tt}\frac{dt}{d\tau} = -\left(1 - \frac{r_s}{r}\right)\frac{dt}{d\tau}\,,$$
$$L = g_{\phi\phi}\frac{d\phi}{d\tau} = r^2\frac{d\phi}{d\tau}\,.$$

as associated conserved quantities. Computing $-d\tau^2(\partial_\tau, \partial_\tau)$ gives

$$-1 = -\left(1 - \frac{r_s}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{1-\frac{r_s}{r}}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2$$

Putting in the conserved quantities found in previous section one gets

$$-1 = -\frac{E^2}{1-\frac{r_s}{r}} + \frac{1}{1-\frac{r_s}{r}}\left(\frac{dr}{d\tau}\right)^2 + \frac{L^2}{r^2}$$

And solving that for $\frac{dr}{d\tau}$ provides

$$\left(\frac{dr}{d\tau}\right)^2 = (E^2-1) + \frac{r_s}{r} - \frac{L^2}{r^2} + \frac{r_s L^2}{r^3} \tag{6}$$

Comparing with the Newtonian case, we can identify a new cubic term and deduce the following relations

$$(E^2-1) = \frac{2\mathcal{E}}{m}, \qquad L = \frac{\mathcal{L}}{m}, \tag{7}$$

so that roughly E and L are the energy and angular momentum in units of the mass of the planet.

Changing variables as before, we obtain

$$\frac{d^2u}{d\phi^2} + u = \frac{r_s}{2L^2} + \frac{3r_s}{2}u^2 . \tag{8}$$

This equation differs from its Newtonian counterpart by the term $\frac{3r_s}{2}u^2$.

(c) In the Newtonian case, we had found

$$\frac{d^2u}{d\phi^2} + u = \frac{r_s}{2L^2} . \tag{9}$$

Plugging in the suggested solution, we find

$$u_0 = \frac{r_s}{2L^2} . \tag{10}$$

The equation $u(\phi) = u_0(1 + e\cos(\phi))$ describes ellipses. To get a more physical intuition, consider

$$r(\phi) = \frac{r_0}{1+e\cos\phi} \tag{11}$$

and its derivative

$$\frac{dr}{d\phi}(\phi) = \frac{er_0\sin\phi}{(1+e\cos\phi)^2} . \tag{12}$$

Clearly, if $e \geq 0$, $\phi = 0$ corresponds to the perihelion, and $\phi = \pi$ is the aphelion.[1] Moreover, if $e < 1$, then there is no point at which $r(\phi)$ diverges, which means the trajectory is a bound orbit. Perihelion and aphelion are respectively at

$$r_{\text{min}} = \frac{r_0}{1+e}, \qquad r_{\text{max}} = \frac{r_0}{1-e} . \tag{13}$$

The inverse relations are

$$r_0 = \frac{2r_{\text{max}}r_{\text{min}}}{r_{\text{max}}+r_{\text{min}}}, \qquad e = \frac{r_{\text{max}}-r_{\text{min}}}{r_{\text{max}}+r_{\text{min}}} . \tag{14}$$

[1]To get the right geometric intuition it is important to keep in mind that in the coordinates we have chosen, $r = 0$ corresponds to the focus of the ellipse occupied by the Sun.

(d) Let us start by estimating the average of u over the orbit

$$\bar{u} = \frac{1}{2\pi}\int_0^{2\pi} d\phi \; u_0(1 + e\cos(\phi)) = u_0 \,. \tag{15}$$

Then, we want to linearize

$$\frac{d^2u}{d\phi^2} + u = \frac{r_s}{2L^2} + \frac{3r_s}{2}u^2 \,. \tag{16}$$

That means we take $\frac{u-\bar{u}}{\bar{u}}$ to be a small dimensionless number and Taylor expand

$$u^2 = \bar{u}^2\left[1 + \left(\frac{u-\bar{u}}{\bar{u}}\right)\right]^2 \approx \bar{u}^2 + 2\bar{u}(u-\bar{u}) = 2\bar{u}u - \bar{u}^2 \,. \tag{17}$$

The equation becomes

$$\frac{d^2u}{d\phi^2} + u = \frac{r_s}{2L^2} + \frac{3r_s}{2}(2\bar{u}u - \bar{u}^2) \,. \tag{18}$$

Reshuffling terms,

$$\frac{d^2u}{d\phi^2} + K^2 u = \tilde{L} \tag{19}$$

with

$$K^2 \equiv 1 - 3r_s\bar{u} \,, \qquad \tilde{L} \equiv \frac{r_s}{2}\left(\frac{1}{L^2} - 3\bar{u}^2\right) \,. \tag{20}$$

It is then immediate to show that

$$u(\phi) = u_0(1 + e\cos(K\phi)) \,, \tag{21}$$

is a solution.

(e) The period of $r(\phi)$ has now changed. The new period is

$$T = \frac{2\pi}{K} \tag{22}$$

Geometrically, that means the ellipse is rotating after each orbit. This is a general property of approximately circular orbits in GR. The precession per orbit is then

$$\Delta\phi = T - 2\pi = 2\pi\left(\frac{1}{K} - 1\right) \approx 3\pi r_s\bar{u} \,, \tag{23}$$

where we have kept only the leading term. That is a good approximation because, even without plugging any numbers, we can tell that Mercury is, infact, very far from the Schwarzschild radius of the Sun.

(f) Restoring natural units and substituting the given data about Mercury's orbit, we obtain

$$\Delta\phi \approx \frac{6\pi GM\bar{u}}{c^2} \approx 3.03 \times 10^{-7} \text{ rad} \tag{24}$$

To turn this into arcseconds per century, we need to use the fact that the orbit of Mercury lasts 88 days. Moreover, one arcsecond is $\frac{1}{3600}$ of a degree, or 4.848×10^{-6} radians. Finally, we use that 88 days is 0.00241 centuries. In total, we obtain

$$\Delta\phi \approx 43''/\text{century} \tag{25}$$

which is the classic result.

2. Light Deflection in General Relativity

(a) The vectors ∂_t and ∂_ϕ are Killing. Therefore the quantities $\dot{x}^\mu K_\mu$ are conserved on geodesics. For these Killing vectors we get

$$E = f(r)\dot{t}, \qquad L = r^2\dot{\phi}, \tag{26}$$

where $f(r) \equiv 1 - r_s/r$. For light-like motions we have the additional constraint

$$0 = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \tag{27}$$

which gives the equations of motion after eliminating $\dot{t}$ and $\dot{\phi}$ using the conserved quantities.

(b) We have

$$\frac{dr}{d\lambda} = \frac{dr}{d\phi}\frac{d\phi}{d\lambda} = \frac{dr}{d\phi}\frac{L}{r^2}. \tag{28}$$

The equations of motion are then

$$E^2 = \left(\frac{dr}{d\phi}\right)^2 \frac{L^2}{r^4} + f(r)\frac{L^2}{r^2.} \tag{29}$$

This can easily be integrated using the separation of variables

$$d\phi\sqrt{E^2 - f(r)\frac{L^2}{r^2}} = \pm dr\frac{L}{r^2}. \tag{30}$$

We are looking for solutions coming from infinity, r is decreasing, and we conventionally choose ϕ to increase, therefore we pick the branch with the minus sign. Integrating we obtain

$$\phi(r) = \frac{L}{E}\int_r^\infty \frac{dr}{r^2}\frac{1}{\sqrt{1 - \frac{f(r)}{r^2}\frac{L^2}{E^2}}}. \tag{31}$$

(c) The point of closest approach is by definition a minimum, so it is reached when $dr/d\phi = 0$. Therefore the constraint reads

$$E^2 = f(r_0)\frac{L^2}{r_0^2} \Leftrightarrow \frac{L}{E} = \frac{r_0}{\sqrt{f(r_0)}}. \tag{32}$$

Substituting this into the integral, we have

$$\phi(r) = \frac{r_0}{\sqrt{f(r_0)}}\int_r^\infty \frac{dr}{r^2}\frac{1}{\sqrt{1 - \frac{f(r)}{f(r_0)}\frac{r_0^2}{r^2}}} \tag{33}$$

By time-reversal symmetry, the point of closest approach is reached at the middle of the photon's journey. Therefore the total angle swiped by the photon is $2\phi(r_0)$. The angle when there is no deflection is π. Therefore the deflection is

$$\Delta\phi = \pi - 2\phi(r_0). \tag{34}$$

(d) To expand the integral it is good habit to first write it in terms of dimensionless parameters. Therefore we introduce $\rho = r/r_0$ such that

$$\phi(r_0) = \frac{1}{\sqrt{1-\frac{r_s}{r_0}}} \int_1^\infty \frac{d\rho}{\rho^2} \left(1 - \frac{1}{\rho^2} \frac{1-\frac{r_s}{r_0\rho}}{1-\frac{r_s}{r_0}}\right)^{-1/2} . \tag{35}$$

We can now expand in r_s/r_0. The first term is just obtained by plugging in $r_s = 0$ and reads

$$\phi(r_0)^{(0)} = \int_1^\infty \frac{d\rho}{\rho^2} \frac{1}{\sqrt{1-1/\rho^2}} = \frac{\pi}{2}, \tag{36}$$

such that the deflection at leading order is $\Delta\phi^{(0)} = 0$.

(e) To go to the subleading order we need the first correction when expanding in r_s/r_0. We get

$$\phi(r_0)^{(1)} = \frac{1}{2}\frac{r_s}{r_0} \int_1^\infty d\rho \frac{1+\rho+\rho^2}{\rho^2(1+\rho)\sqrt{\rho^2-1}} = \frac{r_s}{r_0}, \tag{37}$$

such that the first correction to the deflection angle is

$$\Delta\phi^{(1)} = 2\frac{r_s}{r_0} = \frac{4GM}{r_0}. \tag{38}$$